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Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ (°C)	100	93	70	62	55

A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius (°C), of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

(a) Estimate $T'(7)$. Show the work that leads to your answer. Indicate units of measure.

(b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

(c) Find $\int_0^8 T'(x)\,dx$, and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)\,dx$ in terms of the temperature of the wire.

(d) Are the data in the table consistent with the assertion that $T''(x) > 0$ for every x in the interval $0 < x < 8$? Explain your answer.

(a) $\dfrac{T(8) - T(6)}{8 - 6} = \dfrac{55 - 62}{2} = -\dfrac{7}{2}$ °C/cm

(b) $\dfrac{1}{8}\int_0^8 T(x)\,dx$

Trapezoidal approximation for $\int_0^8 T(x)\,dx$:

$$A = \frac{100 + 93}{2} \cdot 1 + \frac{93 + 70}{2} \cdot 4 + \frac{70 + 62}{2} \cdot 1 + \frac{62 + 55}{2} \cdot 2$$

$$\text{Average temperature} = \frac{1}{8}A = 75.6875°\text{C}$$

(c) $\int_0^8 T'(x)\,dx = T(8) - T(0) = 55 - 100 = -45°\text{C}$

The temperature drops 45°C from the heated end of the wire to the other end of the wire.

(d) Average rate of change of temperature on $[1, 5]$ is $\dfrac{70 - 93}{5 - 1} = -5.75$.

Average rate of change of temperature on $[5, 6]$ is $\dfrac{62 - 70}{6 - 5} = -8$.

No. By the MVT, $T'(c_1) = -5.75$ for some c_1 in the interval $(1, 5)$ and $T'(c_2) = -8$ for some c_2 in the interval $(5, 6)$. It follows that T' must decrease somewhere in the interval (c_1, c_2). Therefore T'' is not positive for every x in $[0, 8]$.

Units of °C/cm in (a), and °C in (b) and (c)

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t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \leq t \leq 80$ seconds, as shown in the table above.

(a) Find the average acceleration of rocket A over the time interval $0 \leq t \leq 80$ seconds. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

(c) Rocket B is launched upward with an acceleration of $a(t) = \dfrac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

(a) Average acceleration of rocket A is

$$\frac{v(80) - v(0)}{80 - 0} = \frac{49 - 5}{80} = \frac{11}{20} \text{ ft/sec}^2$$

(b) Since the velocity is positive, $\int_{10}^{70} v(t)\,dt$ represents the distance, in feet, traveled by rocket A from $t = 10$ seconds to $t = 70$ seconds.

A midpoint Riemann sum is

$$20[v(20) + v(40) + v(60)]$$
$$= 20[22 + 35 + 44] = 2020 \text{ ft}$$

(c) Let $v_B(t)$ be the velocity of rocket B at time t.

$$v_B(t) = \int \frac{3}{\sqrt{t+1}}\,dt = 6\sqrt{t+1} + C$$

$$2 = v_B(0) = 6 + C$$
$$v_B(t) = 6\sqrt{t+1} - 4$$
$$v_B(80) = 50 > 49 = v(80)$$

Rocket B is traveling faster at time $t = 80$ seconds.

Units of ft/sec^2 in (a) and ft in (b)

•

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

Concert tickets went on sale at noon ($t = 0$) and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time t is modeled by a twice-differentiable function L for $0 \le t \le 9$. Values of $L(t)$ at various times t are shown in the table above.

(a) Use the data in the table to estimate the rate at which the number of people waiting in line was changing at 5:30 P.M. ($t = 5.5$). Show the computations that lead to your answer. Indicate units of measure.

(b) Use a trapezoidal sum with three subintervals to estimate the average number of people waiting in line during the first 4 hours that tickets were on sale.

(c) For $0 \le t \le 9$, what is the fewest number of times at which $L'(t)$ must equal 0? Give a reason for your answer.

(d) The rate at which tickets were sold for $0 \le t \le 9$ is modeled by $r(t) = 550te^{-t/2}$ tickets per hour. Based on the model, how many tickets were sold by 3 P.M. ($t = 3$), to the nearest whole number?

(a) $L'(5.5) = \dfrac{L(7) - L(4)}{7 - 4} = \dfrac{150 - 126}{3} = 8$ people per hour

(b) The average number of people waiting in line during the first 4 hours is approximately

$$\frac{1}{4}\left(\frac{L(0) + L(1)}{2}(1 - 0) + \frac{L(1) + L(3)}{2}(3 - 1) + \frac{L(3) + L(4)}{2}(4 - 3)\right)$$

$$= 155.25 \text{ people}$$

(c) L is differentiable on $[0, 9]$ so the Mean Value Theorem implies $L'(t) > 0$ for some t in $(1, 3)$ and some t in $(4, 7)$. Similarly, $L'(t) < 0$ for some t in $(3, 4)$ and some t in $(7, 8)$. Then, since L' is continuous on $[0, 9]$, the Intermediate Value Theorem implies that $L'(t) = 0$ for at least three values of t in $[0, 9]$.

OR

The continuity of L on $[1, 4]$ implies that L attains a maximum value there. Since $L(3) > L(1)$ and $L(3) > L(4)$, this maximum occurs on $(1, 4)$. Similarly, L attains a minimum on $(3, 7)$ and a maximum on $(4, 8)$. L is differentiable, so $L'(t) = 0$ at each relative extreme point on $(0, 9)$. Therefore $L'(t) = 0$ for at least three values of t in $[0, 9]$.

[Note: There is a function L that satisfies the given conditions with $L'(t) = 0$ for exactly three values of t.]

(d) $\int_0^3 r(t)\, dt = 972.784$

There were approximately 973 tickets sold by 3 P.M.

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x	2	3	5	8	13
$f(x)$	1	4	-2	3	6

Let f be a function that is twice differentiable for all real numbers. The table above gives values of f for selected points in the closed interval $2 \leq x \leq 13$.

(a) Estimate $f''(4)$. Show the work that leads to your answer.

(b) Evaluate $\int_2^{13} (3 - 5f'(x))\, dx$. Show the work that leads to your answer.

(c) Use a left Riemann sum with subintervals indicated by the data in the table to approximate $\int_2^{13} f(x)\, dx$. Show the work that leads to your answer.

(d) Suppose $f'(5) = 3$ and $f''(x) < 0$ for all x in the closed interval $5 \leq x \leq 8$. Use the line tangent to the graph of f at $x = 5$ to show that $f(7) \leq 4$. Use the secant line for the graph of f on $5 \leq x \leq 8$ to show that $f(7) \geq \dfrac{4}{3}$.

(a) $f'(4) = \frac{f(5)-f(3)}{5-3} = -3$

(b) $$\int_2^{13} (3 - 5f'(x))\, dx = \int_2^{13} 3\, dx - 5 \int_2^{13} f'(x)\, dx$$
$$= 3(13-2) - 5(f(13) - f(2)) = 8$$

(c) $$\int_2^{13} f(x)\, dx = f(2)(3-2) + f(3)(5-3)$$
$$+ f(5)(8-5) + f(8)(13-8) = 18$$

(d) An equation for the tangent line is $y = -2 + 3(x-5)$.
Since $f''(x) < 0$ for all x in the interval $5 \leq x \leq 8$, the line tangent to the graph of $y = f(x)$ at $x = 5$ lies above the graph for all x in the interval $5 < x \leq 8$.
Therefore, $f(7) \leq -2 + 3 \cdot 2 = 4$.

An equation for the secant line is $y = -2 + \frac{5}{3}(x-5)$.
Since $f''(x) < 0$ for all x in the interval $5 \leq x \leq 8$, the secant line connecting $(5, f(5))$ and $(8, f(8))$ lies below the graph of $y = f(x)$ for all x in the interval $5 < x < 8$.
Therefore, $f(7) \geq -2 + \frac{5}{3} \cdot 2 = \frac{4}{3}$.